Lecture 19

Monday, November 18, 2019 5:48 AM

<u>We summarize</u>: Let f be analytic in $G \subseteq \mathbb{C}$.

① (Baby CIF) If $\overline{B(a,r)} \subset G$, then

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta - a| = r} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in B(a,r)$$

② (Power series rep.) If $B(a,R) \subseteq G$, then $\exists$ unique power series w/ R.O.C. $\geq R$ s.t.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \quad z \in B(a,R).$$

Thus, f is infinitely $\mathbb{C}$-diff. and $a_n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{1}{2\pi i} \displaystyle\int_{|\zeta-a|=r} \frac{f(\zeta) d\zeta}{(\zeta - a)^{n+1}}, \quad r < R.$

• [Do Cauchy Est., $\exists$ primitives, Baby Cauchy from Lecture 18 notes].

<u>Zero set of analytic functions</u>.

① <u>Polynomials</u>. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$ be a (holomorphic) polynomial. We recall from basic algebra the division algorithm: If $q(z)$ is another polynomial, then $\exists$ polynomials $s(z), r(z)$ s.t.

$$p(z) = s(z) q(z) + r(z), \quad \deg r < \deg q.$$

In particular, if $p(a) = 0$, using $q(z) = z - a \Rightarrow p(z) = s_1(z)(z-a)$.

We note $p'(z) = s_1'(z)(z-a) + s_1(z)$. Thus, if also $p'(a) = 0 \Rightarrow s_1(a) = 0$

$\Rightarrow s_1(z) = s_2(z)(z-a) \Rightarrow p(z) = s_2(z)(z-a)^2$. Since $\deg s_1 = \deg p - 1$,

$\deg s_2 = \deg p - 2$, etc., $\exists\, m$ s.t. $p(z) = s_m(z)(z-a)^m$, $s_m(a) \neq 0$

<u>Def</u>. m is the <u>multiplicity</u> of the zero $z = a$ of $p(z)$.

• If $a_1, \dots, a_n \in \mathbb{C}$ are the zeros of p, $m_1, \dots, m_n \in \mathbb{N}$ their multi's, then $p(z) = s_0(z)(z-a_1)^{m_1} \dots (z-a_k)^{m_k}$, $\sum_{j=1}^{n} m_n \leq \deg p$. (1)

<u>Fundamental Thm of Algebra</u> (FTA). Every nonconstant polynomial ($\deg \geq 1$)

Fundamental Thm of Algebra. Every nonconstant polynomial ($\deg \geq 1$) has a zero in $\mathbb{C}$.

Cor. In (1), $\sum_{j=1}^{k} m_k = \deg p$ and $s_0(z) = c$ for some $c \in \mathbb{C}$.

For pf of FTA, we need important result in complex analysis:

Liouville's Thm. Let f be analytic in $\mathbb{C}$ (Def. entire).
If f is bounded ($|f(z)| \leq M$, $\forall z \in \mathbb{C}$), then f is constant.

Pf of LT: Note that f is analytic in $B(a,R)$ for every $a \in \mathbb{C}$, $R > 0$.
By Cauchy's Estimates, $|f'(a)| \leq \frac{M}{R} \to 0 \Rightarrow f'(a) = 0$, $\forall a \in \mathbb{C}$.
But then f is constant as claimed.

$\square$

Pf of FTA. Suppose $p(z) = a_n z^n + \dots + a_0$, $a_n \neq 0$ is a polynomial without zero in $\mathbb{C}$. Then $f(z) = \frac{1}{p(z)}$ is entire. Moreover, $|p(z)| \geq |z|^n \left| |a_n| - \frac{|a_{n-1}|}{|z|} - \dots - \frac{|a_0|}{|z|^n} \right|$

$\Rightarrow |p(z)| \to \infty$ as $|z| \to \infty \Rightarrow |f(z)| \to 0$ as $|z| \to \infty$.

In particular, $|f(z)| \leq 1$ for $|z| \geq R$. If we set $M = \max\left(1, \max_{|z| \leq R} |f(z)|\right) < \infty$, then $|f(z)| \leq M$ in $\mathbb{C} \Rightarrow$
f constant by LT, and hence $p(z)$ is constant.

$\square$.